Analyzing Dynamic Response of Structures with Gamma Distributed Damping

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Abstract—This article presents the main results of a numerical investigation on the uncertainty of dynamic response of structures with statistically correlated random damping Gamma distributed. A computational method based on a Linear Statistical Model (LSM) is implemented to predict second order statistics for the response of a typical industrial building structure. The significance of random damping with correlated parameters and its implications on the sensitivity of structural peak response in the neighborhood of a resonant frequency are discussed in light of considerable ranges of damping uncertainties and correlation coefficients. The results are compared to those generated using Monte Carlo simulation techniques. The numerical results obtained show the importance of damping uncertainty and statistical correlation of damping coefficients when obtaining accurate probabilistic estimates of dynamic response of structures. Furthermore, the effectiveness of the LSM model to efficiently predict uncertainty propagation for structural dynamic problems with correlated damping parameters is demonstrated.

Keywords—Correlated random damping, linear statistical model, Monte Carlo simulation, uncertainty of dynamic response.

I. INTRODUCTION

DAMPING is an important factor in the response analysis and design of dynamically sensitive structures. This has been commonly addressed in structural design through the deterministic use of average damping values without specific attention to the uncertainty associated with this fundamental input parameter. However, for excitation frequencies close to the resonant frequencies, the sensitivity of structural response to damping becomes critical. In addition, the uncertainty and statistical correlation of damping parameters are often unavoidable complications.

The uncertainty in prediction of damping in structural dynamics poses a serious problem to the structural analyst because damping, a parameter of paramount importance for the response and design of dynamically sensitive structures, does not refer unlike other system parameters to a single physical phenomenon, and may depend on a wide range of factors [1]-[3] including vibration amplitudes, nature of structural resisting systems, nature of underlying damping mechanisms, etc. This difficulty has been commonly circumvented in structural design through the deterministic use of an average damping value to characterize the structural capacity of a dynamic system to dissipate energy, without much attention to the uncertainty in this key input parameter.

The difficulty of an accurate prediction of damping emphasizes the uncertainty involved in assessing the dynamic response of structures. For excitation frequencies close to the resonant frequencies the sensitivity of dynamic response of MDOF structures to damping becomes critical. Errors in the estimation of damping matrix will generally result in large errors in the response. The statistical correlation of damping coefficients and the coupling effect of damping in the equations of motion are often unavoidable complications. This problem is of particular importance in relation to the dynamic response analysis of tall structures that rely on damping for their performance under wind [1] and earthquake ground motions [4].

In order to provide additional information for practical applications in engineering design, this paper show the influence of correlated damping parameters on the uncertainty of structural dynamic response. A computational method based on a linear statistical model is implemented to efficiently propagate damping through dynamic analyses to predict second order statistics for the response of MDOF structures with correlated damping parameters. The results are compared to results generated using Monte Carlo simulation techniques. The effect of variability of correlated damping parameters on the dynamic response in the neighborhood of a resonant frequency are presented for considerable ranges of damping uncertainties. In addition, the impact of statistical correlation of damping on the limits of approximation of both the statistical second moment model and the Monte Carlo simulation technique is investigated.

II. UNCERTAINTY PROPAGATION

In the present study the input random variables are **c** designated as $\mathbf{c}=[c_1,...,c_n]^T$. With mean value $\bar{\mathbf{c}}=[\bar{c}_1,...,\bar{c}_2]^T$, and standard deviations $\sigma_c=[\sigma_{c1},...,\sigma_{cn}]^T$. The response output function, **X**=**X**(**c**), is a function of the input random variable **c**, i.e. ***X*** is random field (e.g. [5])

Uncertainties in damping coefficients are propagated through the functional relationships that relate them to the response for determining the uncertainty of dynamic response estimates. Thus, to implement uncertainty of damping in the dynamic analysis recall that, the matrix system of differential equation of motion governing the displacement ***X***(t) an nMDOF discretized dynamic system subjected to an external excitation ***F***(t) can be written as:

$$[M]\ddot{X}+[C]\dot{X}+[K]X=F(t) \quad (1)$$

where ***F*** is general function of time.

A physical system described by (1) and representative of a typical industrial building with rigid floors is shown in Fig. 1, for n=5. In this system $[M]$ and $[K]$ are deterministic: The damping matrix $[C]$ depends linearly on the viscous damping coefficients c_i according to the definition of its elements (e.g. [6]).

$$C_{ij} = \begin{cases} c_i + c_{i+1} & \text{si} \quad i\text{-}j=0 \\ -c_i & \text{si} \quad i\text{-}j=1 \\ -c_j & \text{si} \quad j\text{-}i=1 \\ 0 & \text{si} \quad |i\text{-}j|>0 \end{cases}$$

In this work uncertainty in damping, represented by the statistically correlated, randomly distributed damping coefficients according to Gamma distribution, is propagated using Monte Carlo simulation techniques and a statistical linear with correlated parameters based on sensitivity derivatives.

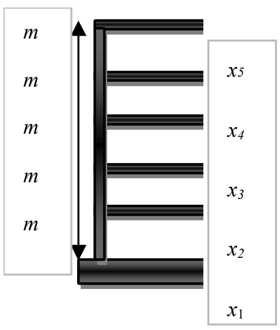


Fig. 1 Five story example building with random damping characteristic ($\bar{c}_i$,σ_{ci} and ρ_{cij} ;(i=1,..,n)

A. Definition and Properties of the Gamma Distribution

We say c is gamma-distributed with parameters k and λ, if their density function has the form in (2):

$k-1$ $-\lambda c$

variables. More generally the parameter k need not be integervalued, when the gamma distribution is written:

$$f_{ck}(c) = \frac{\lambda(\lambda c)^{k-1} e^{-\lambda c}}{\Gamma(k)}, \ c \geq 0 \qquad (4)$$

The only restrictions are λ>0 and k >0. The gamma function Γ(k) is equal to (k-1)! if k is an integer, but more generally is defined by definite integral

$$\Gamma(k) = \int_0^{\infty} e^{-u} u^{k-1} du \qquad (5)$$

The integral arises here as a constant needed to normalize the function to proper density function.

B. Monte Carlo Simulation

This technique consists essentially in generating numerically statistical results of the response without performing any physical experimentation. The computer simulation involves sampling at random to simulate numerically large number of experiments and to observe the results. In the present case, sequence of correlated damping coefficients (N) Gamma- distributed with prescribed mean and standard deviation is first generated. The dynamic system is then solved for each value of the random variable c_i to give a sample value of response X_i. These sample values are finally used to determinate the second order statistics of the response as:

$$\overline{X}(c) = \frac{1}{N}\sum_{i=1}^{N} X(c_i), \quad (i=1,2,\ldots..,N) \qquad (6)$$

$$\sigma^2 X = \frac{\sum_{i=1}^{N} (X(c_i) - \overline{X}(c))}{N-1} \qquad (7)$$

The problem with a traditional MC simulation is that in order to get an accurate prediction of output mean and variance one may have to perform thousands of runs. The computation is very long and expensive (1) must be solved a number of times equal to **N** the number of iterations.

c_k $-1)!$ $(k$

By integration or, by consideration of c as the sum of k independent exponentially distributed random variables, then:

$$\bar{c} = \frac{k}{\lambda} \text{ and } \sigma_c^2 = \frac{k}{\lambda^2} \quad (3)$$

In fact, the gamma distribution is more broadly defined than is implied by its derivation as the distribution of sum k independently, identically distributed exponential random

$$f(c) = \lambda \underline{\qquad\qquad} (\lambda c)\ e, \quad c \geq 0 \quad (2)$$

The damping coefficients c_i are assumed to be Gamma-distributed with mean value $\bar{e}_i$ and standard deviation. The validity of the gamma distribution of damping and its probabilistic characteristics based on full scale measurement of buildings has been checked by [7]. Consider X as a function of **c** only and denote by X^0 the value of X when **c** takes on its mean value $\bar{e}$. Then the vector X can be expanded in a Taylor series about **c**= **e** as:

$$X(c) = X(e) + \sum_{i=1}^{n} \frac{\partial X}{\partial c_i} (c_i - e_i) \quad (8)$$

C. Linear Statistical Model (LSM)

where; the higher order terms have been neglected in the present case.

Introducing sensitivity derivatives;

$$\xi_{ij} = (\partial X_i / \partial c_j)_{c_k = e_k}, \text{ for } \quad k=1\ldots n \quad (9)$$

It can be shown (e.g., [8]) that for any distribution of input variables, the approximations given by the (10) and (11) for the mean and variance of the output function X hold:

$$\bar{X}(c) = X(e) \quad (10)$$

$$Var[X_i] = \sum_i^n \sum_j^n \left.\frac{dX_i}{dc_i}\right|_{e_i} \left.\frac{dX_i}{dc_j}\right|_{e_j} COV[c_i, c_j] \quad (11)$$

where; $COV[c_i, c_j]$ is the covariance matrix of the pair of random damping variables c_i and c_j . Its ij[th] element is defined by $COV_{ij} = \rho_{ij}\sigma_{ci}\sigma_{cj}$ where $\rho_{ij} = 1$ if i=j.

The elements of the main diagonal are the variance and are known from the assumed distribution law. The off diagonal elements contain the correlation ρ_{ij} between the various damping coefficients.

Note that when the correlation coefficients $\rho_{ij} = 0$ for i≠j, the input random variables **c** are said to be uncorrelated whereas when $\rho_{ij} = 1$ for i≠j, they are said to be fully correlated.

Various assumptions can be made regarding the partial correlations ρ_{ij} between the various damping coefficients.

However, we prefer to give this problem which is random in nature, a random solution. Once the variances and correlations of the damping coefficients are prescribed, a Monte Carlo simulation of damping coefficients generates a Gamma distribution and a covariance matrix whose RMS of the offdiagonal elements converge to the RMS of prescribed values. The off-diagonal terms of the matrix thus generated are assumed to represent the correlation in the system. Note that the computer simulation of damping coefficients using the Monte Carlo simulation technique is here based on a multivariate Gamma distribution with prescribed mean vector $\bar{e} = [e_1, \ldots, e_n]^T$ and covariance matrix COV_{ij}. While the multivariate Gamma cumulative distribution function is not trivial to compute in high dimensions [9], it is available in commonly used software packages such as Matlab.

The sensitivity functions ξ_{ij} are available by differentiation of (1) with respect to c_j as:

$$[\,]$$

The left side of (13) is identical to that of (1), and the right side can be interpreted as a fictitious forcing vector.

In the case of general loading $F(t)$, the right side of (13), can be obtained from the time derivative $\dot{X}$ of the response X^0, the value of X when **c** takes on its mean value $\bar{e}$. For systems with large numbers of degrees of freedom, the vector X^0 can be computed by solving (1) with nominal damping $c_j = \bar{e}_j$ using mode superposition analysis. Alternatively, the vectors X and $\dot{X}$ can be obtained systematically by using step by step integration methods of structural dynamics (e.g. [6]). Thus, to obtain the vectors ξ_j (j=1 …n), (1) must be solved first with the forcing vector ***F(t)*** after which (13) is solved, n times.

Thus, the global bulk of computations involve essentially, a computer simulation for the evaluation of the covariance matrix [*COV*] then, the solution of (1) for X^0, $\dot{X}^0$ and finally the solution for ξ_j, n times.

III. Numerical Example

A numerical example utilizing a typical industrial building, modeled as a lumped mass system, is considered to predict uncertainty propagation and examine quantitatively the influence of uncertainty level in the damping on the overall response under dynamic excitation induced by a rotating machine.

The structural system shown in Fig. 1 is taken as basic model for the computations. For the sake of clarity, the lumped mass at each floor, the inter-story stiffness and the mean value of inter-story damping constant between each level are kept constant in this study, although different values could be used as input. The lumped mass at each floor is equal to $m_1= m_2= m_3= m_4= m_5=150$t and the inter-story stiffness between each level is such that $k_1= k_2= k_3= k_4= k_5=210*10^3$ KN/m. The mean values $\bar{c}$ of the inter-story damping constants, examined in this study were (in ascending order) $\bar{c}$ =394.37, $\bar{c}$=788.74, $\bar{c}$=1183.11, $\bar{c}$=1577.48 and $\bar{c}$=1971.86 KN/m/s respectively (in the example considered, these values correspond to 1%, 2%, 3%, 4%, 5% of critical damping respectively). Each value of the damping constant was assigned a Cov that varied from 10% to 50% with uniform increments of 10%. The dynamic excitation resulting from a rotating machine, applied at the first story is represented by a harmonic function of amplitude F=1400 KN and frequency Ω=31.09 rad/s corresponding to the second resonant frequency of the structural system.

$$[M]\frac{\partial \ddot{X}}{\partial c_j}+[C]\frac{\partial \dot{X}}{\partial c_j}+[K]\frac{\partial X}{\partial c_j}=-\left(\frac{\partial C}{\partial c_j}\right)\dot{X} \quad (12)$$

or

$$[M]\ddot{\xi}_j+[C]\dot{\xi}_j+[K]\xi_j=-(\partial[C]/\partial c_j)\dot{X} \quad (13)$$

TABLE I
Comparison between Statistical Linear model (LSM) and Monte Carlo Simulation (MCS) Method for Uncorrelated and Perfectly Correlated Damping Parameters

	Standard deviations of (ρc i,j=0)	(ρc i,j=1)	Standard deviations of Mean values of		Error Coefficients of variation	Error	response (ρc i,j=0)		response (ρc i,j=1)	
N°	LSM	MCS	LSM	MCS	%	%	Damping $\bar{c}_i$	$\sigma_{ci}/\bar{c}$	$\sigma_{X_i}/\bar{X}_i$ (SDSL) ρc ij =0	ρc ij =1
1	0,895	1,037	1,717	3,174	13,77	45,91	0,26	0,49		
2	1,169	1,370	2,250	4,177	14,69	46,14	0,26	0,50		
3	0,645	0,749	1,230	2,283	13,88	46,12	394,37	0,5	0,26	0,50
4	0,339	0,387	0,640	1,184	12,35	45,92	0,26	0,49		
5	1,076	1,264	2,068	3,844	14,87	46,19	0,26	0,50		
1	0,359	0,393	0,684	0,978	**8,61**	30,07			0,20	0,15
2	0,466	0,521	0,900	1,315	**10,41**	31,56			0,21	0,15
3	0,265	0,290	0,494	0,720	**8,44**	31,33	788,74	0,4	0,22	0,15
4	0,141	0,146	0,255	0,366	**3,35**	30,16			0,21	0,14
5	0,431 0,474 0,829 1,217 **9,04** 31,89 0,21 0,14 1 0,181 0,185 0,340 0,40 2,42 14,22 0,14 0,10									
2	0,233	0,254	0,450	0,55	8,42	18,72	0,16	0,10		
3	0,139	0,144	0,250	0,30	3,10	17,81	1183,11	0,3	0,17	0,10
4	0,075	0,078	0,127	0,15	4,32	14,65	0,16	0,10		
5	0,217	0,241	0,416	0,52	9,77	19,66			0,16	0,10
1	0,092	0,096	0,169	0,180	4,64	**5,71**			0,09	0,05
2	0,116	0,123	0,225	0,247	5,33	**8,64**			0,10	0,05
3	0,074	0,077	0,127	0,135	3,45	**6,36**	1577,48	0,2	0,12	0,05
4	0,040	0,042	0,063	0,067	5,64	**5,59**			0,11	0,05
5	0,110	0,118	0,209	0,232	7,51	**9,91**			0,11	0,05

1	0,035	0,035	0,065	0,066	1,44	1,90	0,04	0,05		
2	0,046	0,046	0,090	0,092	0,78	1,89	0,05	0,05		
3	0,028	0,028	0,051	0,051	1,35	1,38	1971,85	0,1	0,06	0,05
4	0,014	0,014	0,025	0,026	1,47	1,93	0,04	0,05		
5	0,044	0,045	0,084	0,086	1,62	2,41			0,06	0,05

IV. Sample Results and Discussion

A. Uncertainty Propagation

In Table I, the standard deviations of peak story displacement for several combinations of mean damping values $\bar{c}_i$ and corresponding covariance coefficients are (for the sake of clarity) presented for the two cases of uncorrelated and fully correlated damping parameters. The results obtained from the application of the LSM model are systematically compared with the corresponding results derived from the MCS method. It is to be noted that the two methods are in acceptable agreement up to Cov of damping values less than or equal 40% in the case of uncorrelated damping parameters and up to Cov of damping less than 30% in the case of perfectly correlated damping. Both methods are in excellent agreement up to Cov of damping values less than or equal 10%. It also is observed that the covariance coefficient of peak story displacement response is equal to the covariance coefficients of damping for the case of perfectly correlated damping parameters whereas it is only half this value in the uncorrelated case, regardless of the mean damping values considered.

If the uncertainty about damping is such that larger Cov values should be considered, the LSM becomes inadequate and higher order statistics based on second order sensitivity derivatives should be considered.

In Fig. 2, the standard deviation of building response at the second floor (corresponding herein to the maximum floor displacement) as function of Cov of damping is presented for different values of correlation coefficient. It is seen that the uncertainty in damping influences the system response. Depending on the mean value of damping the effects are more pronounced for light damping, high variability of damping and strong correlation between damping coefficients.

In Fig. 3, the standard deviation of building responses with light damping at the second floor is presented for both LSM and MCS for strong and low correlation. The results suggests that differences in standard deviation of building response obtained for both methods are insignificant for small values of Cov of damping. However, for larger values, the errors introduced by the linearization technique, increase concomitantly with an increase in Cov of damping. Moreover it should be noted that large dispersion in results between the LSM and MCS methods is observed for dynamic systems with light damping, large values of damping variability and strong correlation.

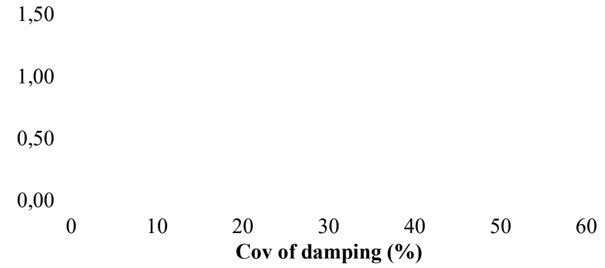


Fig. 2 Standard deviation of response (LSM) at the 2nd floor for different correlation coefficient with light damping ($\bar{c}$=197.19)

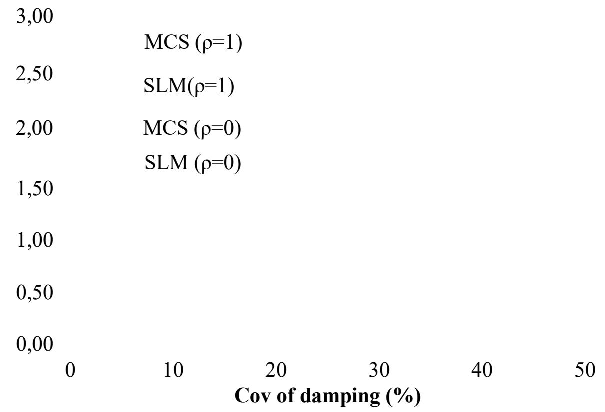


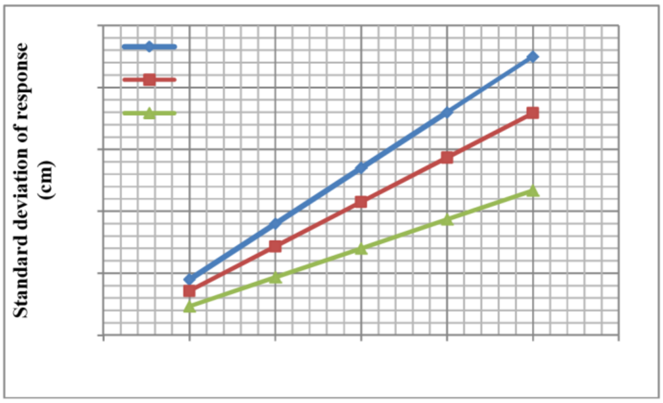
Fig. 3 Comparison Standard deviation of response between LSM and MCS at the second floor of building with light damping

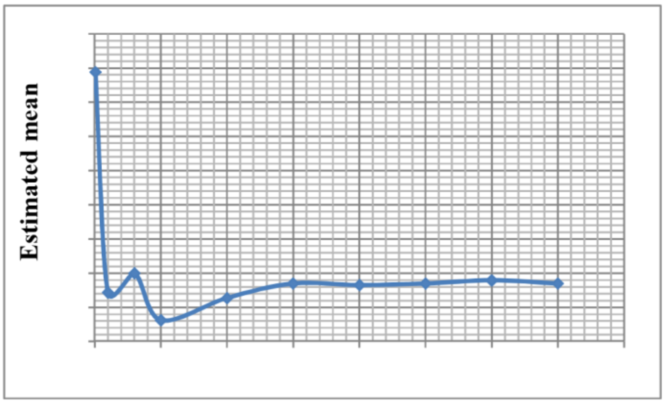
B. Convergence Timing Considerations

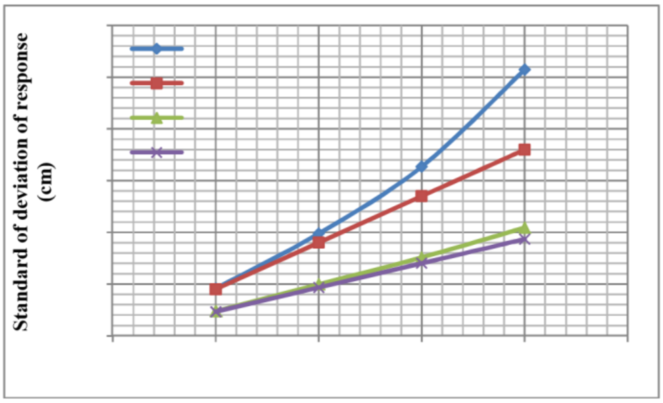
A stochastic analysis with a MC simulation of N runs can be computationally expensive especially for high an n-MDOF dynamic system. In the example presented here, this number was fixed at one thousand five hundred in the case of low correlation [10] while in the case of a strong correlation number is fixed at four thousand in accordance with the progressive results obtained for the estimate of the mean and its estimated variance as a function of the number of samples. Typical convergence of these estimates with increasing sample size is illustrated in Figs. 4 and 5, respectively. In other cases, however, the slow convergence of statistical processes may require even more iterations. The savings in computer time achieved with the linear model become quite evident. In the present study (with five random input variables), the linear statistical model is very efficient; the LSM method requires approximately the computational equivalent of five analysis runs. Note that the timing associated of the LSM model is due to the calculation of first-order sensitivity derivatives function. With the incremental iterative method used herein, the computational expense is dependent on the number of input variables and output functions. An approach for uncertainty propagation and robust design in computational fluid dynamics using sensitivity derivatives can be found in [11].

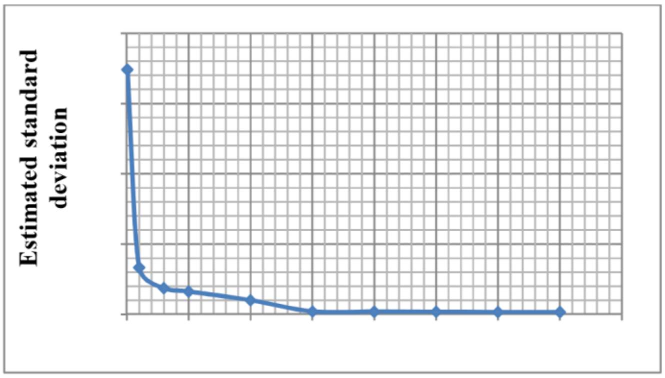
0,3

0,2









4,76

Fig. 4 Convergence of mean estimate with increasing sample size

Fig. 5 Convergence of standard deviation estimate of mean with increasing sample size

0,4

C. Probability Density Function Approximations

Approximating a mean and standard deviation of the response output function ***X*** and assuming a normal distribution, one may then construct a PDF to approximate the behavior of ***X***. Two independent MC simulations with a sample size of N=4000 were conducted. In both simulations, the average values of input parameters were set at $\bar{e}=[e_1,...,e_5]=394,37$ *KN/m/s*. In Simulation 1 $\sigma_c=[5\%e_1,...,5\%e_5]=19,72$ *KN/m/s* while in Simulation 2, $\sigma_c=[10\%e_1,...,10\%e_5]=39,437$ *KN/m/s*. The results are plotted in Figs. 6 and 8 respectively. These approximations are compared to the PDF histograms generated from MC simulations. The bars depict the actual MC simulation histograms and the solid curves represent the normal distributions at the MC mean value and standard deviations. The MC simulation size of 4000 is not sufficient to obtain a smooth marginal PDF for the case described in Fig. 8. It is apparent however, that for small standard deviations of the damping (Fig. 6), the normal marginal PDF approximates the actual simulation in regions about the mean and the tails of the distribution. This is significant, for if one is primarily interested in reliable failure predictions, the prediction may be good enough, and the approximations of 4000 samples may suffice. This unfortunately is not the case for larger standard deviations of damping as clearly indicated in Fig. 8. In such a case, a simulation size of N=6000 would likely produce much better approximations. It also should be noted that the results of MCS method can be utilized for the probabilistic description of response in terms of its second order statistics, as illustrated respectively in Fig. 7 for the approximate PDFs of the damping coefficients and in Fig. 9 for the maximum dynamic response. It is also noted that the despite the small effect of damping correlation on the marginal PDF of damping, significant differences are observed in the corresponding marginal PDF of the response.

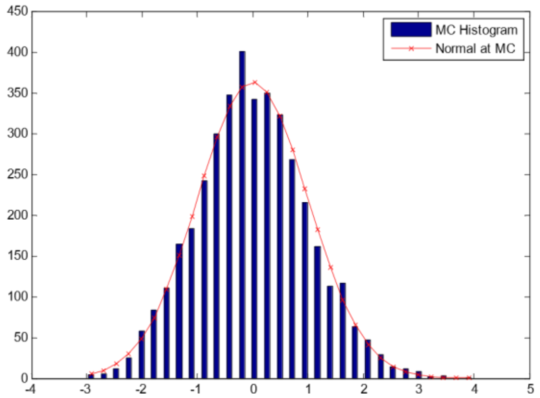


Fig. 6 Histogram and marginal PDF ($\bar{e}_2 = 394,37$ KN/m/s, $Cov_c=0,05$ and $\rho_{c\ i,j}=0$) of peak response at 2nd floor

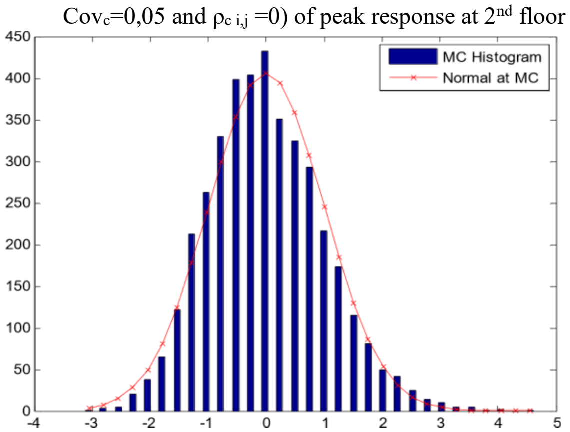


Fig. 8 Histogram and marginal PDF ($\bar{e}_2 = 394,37$ KN/m/s, $Cov_c=0,1$ and $\rho_{c\ i,j}=0$) of peak response at 2nd floor

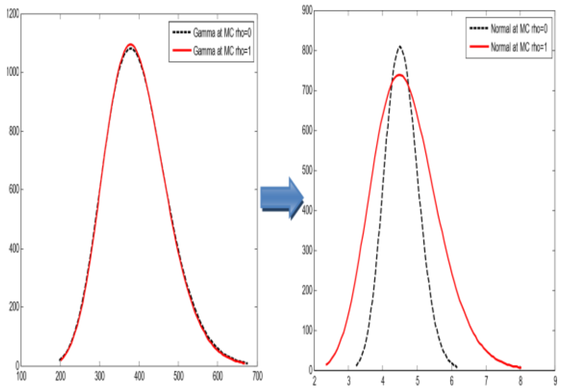
Fig. 7 Marginal PDF of damping (assumed $\bar{e}$=394,3 KN/m/s, Covc=0.2) for two values of correlation coefficients

V. Summary and Conclusions

This paper investigated the significance of damping variability on the dynamic response of typical building structures with statistically correlated damping in the neighborhood of a resonant frequency for various degrees of damping correlation coefficient.

The present results represent an implementation of linear statistical model with correlated parameters for uncertainty propagation to predict second order statistics of peak response for structural dynamics problems. Efficient calculation of sensitivity derivatives was employed and the validity of the model was assessed by comparison with statistical moments generated through independent MC simulations for different degrees of correlation. Excellent agreement has been obtained for damping uncertainties ranging up to Cov of damping values less than or equal 40% and approximately 20% in the

Fig. 9 Marginal PDF response at 2nd floor for the values of



correlation coefficients (computed Cov_{X2}=0.2 for $\rho_{c\ i,j}$ =1 and Cov_{X2}=0.1 for $\rho_{c\ i,j}$=0)

two extreme cases of uncorrelated and perfectly correlated damping coefficient respectively. Collectively, these results demonstrate the possibility for an effective approach to treat input parameters uncertainty and its propagation through structural dynamic analysis without large numbers of samples. Furthermore, the numerical results show the importance of damping uncertainty and statistical correlation of damping coefficients for accurate estimates of dynamic response under resonant conditions.

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