

Quantum Harmonics in Fractal Resonance

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Abstract,Q WKLV SDSHU WKH SUREOHP RI D PL[HG0RGH FUDFN HPEHGGHG LQ DQ LQILQLWH PHGLXP PDGH RI D IXQFWLRQDOO\ JUDGHG SLH]RHOHFWULF PDWHULDO)*30 ZLWK FUDFN VXUIDFHV VXEMHFWHG WR HOHFWURPHFKDQLFDO ORDGLQJV LV LQYHVWLJDWHG (ULQJHQ\IV QRQORFDO WKHRU\ RI HODVWLFLW\ LV DGRSWHG WR IRUPXODWH WKH JRYHUQLQJ HOHFWUR HODVWLFXODVWLRQV 7KH SURSHUWLHV RI WKH SLH]RHOHFWULF PDWHULDO DUH DVVXPHG WR YDU\ H[SRQHQLWDOO\ DORQJ D SHUSHQGLFXODU SODQH WR WKH FUDFN8VLQJ)RXULHUWUDQVIRUPWKUHHLQWHJUDOHTXDWLQV DUHREWDLQHG LQ ZKLFK WKH XQNQRZQ YDULDEOHV DUH WKH MXPSV RI PHFKDQLFDO GLVSODFHPHQWV DQG HOHFWULF SRWHQWLDOV DFURVV WKH FUDFN VXUIDFHV 7R VROYHWKHLQWHJUDOHTXDWLQVWKHXQNQRZQVDUHGLUHFWO\H[SDQGHGDVD VHULHVRI-DRELSRO\QRPLDOVDQGWKHUHVXOWLQJHTXDWLQV VROYHGXLQJ WKH6FKPLGWPHWKRQ,QRQWUDVWWRWKHFODVVLFD OVROXWLRQVEDVHGRQ WKH ORFDO WKHRU\ LW LV IRXQG WKDW QR PHFKDQLFDO VWUHVV DQG HOHFWULF GLVSODFHPHQWVWLQJXODULWLHVDUHSUHVHQWDDWKH FUDFNWLSVZKXQQRORFDO WKHRU\LVHPSOR\HGWR LQYHVWLJDWHWKHSUREOHP\$ GLUHFWEHQHILWLVWKH DELOLW\WRXVHWKHFDODXODWHGPD[LXPVWUHVVD VDIUDFWXUHFULWHULRQ 7KH SULPDU\ REMHFWLYH RI WKLV VWXG\LV WR LQYHVWLJDWH WKH HIIHFW RI FUDFN OHQJWK PDWHULDO JUDGLHQW SDUDPHWHU GHVFULELQJ)*30V DQG ODWWLFHSDUDPHWHURQWKHPHFKDQLFDOVWUHVVD QGHOHFWULFGLVSODFHPHQW ILHOGQHDFUDFNWLSV

Keywords)XQFWLRQDOO\ JUDGHG SLH]RHOHFWULF PDWHULDO PL[HG PRGHFUDFNQRQORFDOWKHRU\6FKPLGWPHWKRQ

,,17528&7,21

,6&29(5(' DV HDUO\ DV E\ 3LHUH DQG -DFTXHV &XULH SLH]RHOHFWULF\ FDQ EH GHILQH DV WKH OLQHDU HOHFWURPHFKDQLFDO LQWHUDFWLRQ EHWZHHQ WKH PHFKDQLFDO DQG HOHFWULFDOVWUHVVRIFU\WVDOVGHYRLGRIDFHQ WHURIV\PPHWU\>@

3LH]RHOHFWULFDPWHULDOVH[KLELWVKHSLH]RHOHFWULFHIIHFWWKDWLV HOHFWULFSRODUL]DWLRQLVLQGXFWGLQWKHPDW HULDORQDSSOLFDDWLQRI PHFKDQLFDO ORDGV DQG YLFHYHUVD >@ +RZHYHU D PDMRU VKRUWFRPLQJ RI SLH]RHOHFWULF FHUPLFV LV WKDW WKH\ DUH H[WUHPHO\EULWVWZLWKDSURSHQVLW\WRGHYH ORSFUDFNVGXHWV VWUHVV FRQFHQWUDWLQV LQGXFWG DV D FRQVHTXHQFH RI ERWK PHFKDQLFDO DQG HOHFWULFDO ORDGLQJV >@ \$ FODVV RI PDWHULDOV ZKLFK FRXOG SRWHQWLDOO\ DOOHYLDWH WKH SUREOHP RI LQWHUQDO GHERQLQJ DQG VWUHVV FRQFHQWUDWLQV LQ FRQYHQWLRQDO SLH]RHOHFWULF PDWHULDOV DUH IXQFWLRQDOO\ JUDGHG PDWHULDOV)*0V 2YHU WKH SDVW WZR GHFDGHV)*0V KDYL DOUHGD\ VKRZQ JUHDW SURPLVH WR EH XVHG DV DQ DOWHUQDWLYH WR FRQYHQWLRQDO KRPRJHQRXV FRDWLQJV >@ 7KHVH PDWHULDOV DUH JHQHUDOO\ FRPSRVHG RI DW OHDVW WZRSDVH LQKRPRJHQRXV

1LGKDO-DPLDLVZLWKWKHSHURVSDFH'HSUWPHQW%D&DPSXV6ZDQV HD 8QLYHUVLW\6ZDQVHD6\$(18.HPDLOQLGKDOMHPDLD#KRWPDLO FRP 6DPL (0%RUJL LV ZLWK WKH 0HFKDQLFDO (QJLQHHULQJ 'HSUWPHQW 7H[DV \$ 08QLYHUVLW\DW4DWDU(QJLQHHULQJ)%XLOGLQJ32% R[(GXFDWLQJ &LW\ 'RKD 4DWDU SKRQH HPDLO VDPLHOBERUJL#TDWUWDPXHGX SDUWLFXODWHFRPSRVLWHV\QWKHVL]HGLQVXFK DPDQQHUWKDWWKH YROXPHIUDFWLRQVRIWKHFRQVWLWXHQWVYDU\F RQWLQXRXVO\DORQJ DQ\ GHVLUHGH VSDWLDQ GLUHFWLRQ UHVXOWLQJ LQ PDWHULDOV KDYLQJ VPRRWK YDULDWLQRI PHFKDQLFDO SURSHUWLHV 7KHUHIRUH DQ\ PDWHULDO SURSHU\ GLVFRQWLQXLWLHV WKDW H[LVW PD\EH VXLWDEO\ HOLPLQDWHG ZKHQ)*0V DUH XVHG DV FRDWLQJ PDWHULDO 7KHUHIRUH WKH FRQFHSW RI)*0V FDQ EH H[WHQGHG WR SLH]RHOHFWULF PDWHULDOV WR V\QWKHVL]H)*30V WKDW H[KLELW LPSURYHG\HOLDELOLW\RUUHOLDEOHVHU\YLFHOL IHWLPHSUHGLFWLRQRI)*30 EDVHG GHYLFHV LW LV

LPSHUDWLYH WR LQYHVWLJDWH DQG XQGHUVVDQG WKHIUDFWXUHEHKDYLRURI)*30VDQ GWKHLUHIIHFV RQ WKH HOHFWURPHFKDQLFDO UHVSQRVH ORVW RI WKH SUHYLRXV LQYHVWLJDWLRQV LQWR WKH IUDFWXUH RI)*30V DGRSW FODVVLFO FRQWLQXXPPHFKDQLFVEDVHGRQWKHORFDOWKHR U\$QRWKHUSRLQW HDVLO\ QRWLFHG LV WKDW WKHVH VWXGLHV DUH DOPRVW H[FOXVLYHO\ GHYRWHGWRWKHDQDO\VLVRIDQWLSODQH0RGH,,F UDFNSUREOHPV &RQVLGHULQJ ERWK HOHFWULFDOO\ LPSHUPHDEOH DQG SHUPHDEOH FUDFN VXUIDFHV &KXH DQG 2X >@ VWXGLHG WKH SUREOHP RI D 0RGH,,FUDFNRLHQWHGSHUSHQGLFXODUWRWKHL QWHUIDFHIRUPHG EVERQGLQJWZRIXQFWRQDOO\JUDGHGSLH]RHOHFW ULFKDOISODQH +VXDQG&KXH>@VROYHGWKH0RGH,,IUDFWXUHSUR EOHPRIDQ DUELWUDULO\RULHQWHGFUDFNLQIXQFWRQDOO\ UDGHGSLH]RHOHFWULF VWULSERQGHGWRDKRPRJHQRXVSLH]RHOHFWULFK DOISODQH&KHQ DQG&KXH>@LQYHVWLJDWHGWKH0RGH,,IUDFWXU HSUREOHPRID FUDFNHGIXQFWRQDOO\JUDGHGSLH]RHOHFWULFV XUIDFHOD\HUERQGHG WR D FUDFNHG IXQFWRQDOO\ JUDGHG SLH]RHOHFWULF VXEVDWUDWH \$ VXUYH\ RI WKH OLWHUDWXUHV VKRZV WKDW DV RSSRVHG WR 0RGH,, FUDFNSUREOHPVLQ)*30VVWXGLHVFRQFHUQLQJ0RG H,0RGH ,,DQGPL[HG0RGHFUDFNSUREOHPV\DUHIDUDQGIHZEH WZHHQ 8HGD>@DQDO\]HGWKHPL[HG0RGHG\QDPLFIUDFWXU HSUREOHP IRUIXQFWRQDOO\JUDGHGSLH]RHOHFWULFVWULSF RQWDLQLQJDSUDOOHO FUDFNXQGHULQSODQPHFKDQLFODQGH0HFWULF LPSDFWORDGLQJV 5HFHQWO\=KRXDQG&KHQ>@H[DPLQHGWKHLQWHU DFWRQRWZR SDUDOOHO 0RGH, OLPLWHGSHUPHDEOH FUDFNV LQ)*30V &RPPRQ WR DOO DIRUHPHQWLRQHG VWXGLHV WKH XVH RI FODVVLFO FRQWLQXXP PHFKDQLFV WHFKQLTXHV LV EDVHG RQ WKH ORFDO DVVXPSWLRQ WR LQYHVWLJDWH FUDFN SUREOHPV LQ)*30V \$FFRUGLQJ WR ORFDO HODVWLFLW\ WKHRU\ WKH VWDWH RI VWUHV DW D VSHFLILF SRLQW LQ WKH PDWHULDO GHSHQGV RQ\ RQ WKH VWDWH RI VWUDLQDWWKHVDPHSRLQW&RQWUDU\WRSK\VLFD OUHDRVRLQJWKH DSSOLFDWLRQRORFDOHODVWLFLW\WKHRU\LQYD ULDEO\OHDGVWRVWUHV V LQJXODULWLHVDWKKHFUDFNWLSV\$PDMRULVX HKHUHLVWKDWVWUHV DW WKH FUDFN WLSV LV

LQGHWHUPLQDWH DQG WKXV WKHUH D IUDFWXUH FULWHULRQ EDVHG RQ PD[LXP VWUHV LV QRW HDV\ WR HVWDEOLVK 'LIIHUHQWIURPFODVVLFOORFDOHODVWLFLW\WK HRU\LVWKHQORFDO HODVWLFLW\ WKHRU\ ZKLFK DWWHPSWV WR GHYHORS WKH FRQVWLWXWLYH UHODWLRQV KLSV ZLWKRXW IRUHJRLQJ WKH PLFURVWUXFWXUH RI WKH PDWHULDO1RQORFDOFRQWLQXXPPHFKDQLFVLQW LDWHGE\ULQJHQ >@LVEDVHGRQWKHQORFDOHODVWLFLW\PRGHO ZKHUHWKHVWDWH RIVWUHVVDWJLYHQSRQLQWLV\DXQFWRQRWKRHV WUDLQVWDWHV\DWDOO SRLQWVLQWKHPDWULDO7KHQRORFDOWKHRU\Z DVHPSOR\HGWR JUHDWVXFFHVVE\ULQJHQ>@WRLQYHVWLJDWHWK HVWUHV\QHDUWKH WLSRIDVKDUSOLQHFUDFNLQDQLVRWURSLFHODVW LFSODWH8QOLNH FODVVLFDORU ORFDO HODVWLFLW\ WKHRU\ LW ZDV VKRZQ WKDW WKH VWUHV ILHOG FDOFXODWHG E\ XVLQJ QRORFDO WKHRU\ GRHV QRW FRQWDLQDQ\VLQJXODULWLHVDWKKHFUDFNWLSV 7KHUHIRUHQRORFDO HODVWLFLW\PRGHOVFDQEHXVHGWRREWDLQWKH VWUHV\ILHOG\QWKH YLFLQLWRIWKHFUDFNWLSVDQGHVWDEOLVKDIUD FWXUHFULWHULRQLQD QDWXUDOZD\$PDMRUKLQGDQFHWRVWXG\QJFUD FNSUREOHPVLQ)*30V YLD QRORFDO HODVWLFLW\ PRGHOV LV WKH LQKHUHQW PDWKHPDWLFDO GLILFXOW\ DVVRFLDWHG ZLWK LW 7R WKH EHVW NQRZOHGJH RI WKH DXWKRUV WKH HOHFWURHODVWLF EHKDYLRU RI)*30V ZLWK D PL[HG0RGH HOHFWULFDOO\ LPSHUPHDEOH FUDFN KDV QRW EHHQ VWXGLHG XVLQJ WKH QRORFDO WKHRU\ LQ RSHQ OLWHUDWXUH 7KHUHIRUH LQ WKLVSDSHU WKH FRQFHSW RIULQJHQ\V QRORFDO WKHRU\ LV DGRSWHG WR VROYH WKH PL[HG0RGH FUDFN SUREOHP(LQD)*307KHPDMRUHPSKDVLVLVWRFDQFXO DWHWKH VWUHVVDQGH0HFWULFILHOGVLQ)*30VZLWKPL[HG 0RGHFUDFNV WKURXJKDIRUPXODWLRQEDVHGRQWKHQORFDO WKHRU\ 7KLVSDSHULVRUJDQL]HGDVIRORZV7KHIRUPXOD WLRQDQGWKH ERXQGDU\FRQGLWLRQVRIWKHSUREOHPDUHSUHVH QWHGLQ6HFWRQ ,,7KHVROXWLRQPHWKRG\LVGHVFULEHGLQ6HF WLRQ,,ZKHU WKHPL[HGERXQGDU\YDOXHSUREOHPVUHGXFHGW RDU\VVHPRV VLQJXODULQWHJUDOHTXDWLRQV6HFWRQ,9FRQW DLQVWKHYDOLGDWLRQ RI UHVXOW

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+HUH s LV WKH)RXULHU WUDQVIRUP YDULDEOH
DQG #+DQG I
UHSUHVHQW WKH WUDQVIRUPHG XQNQRZQ
YDULDEOHV 7KLV
WUDQVIRUPDWLRQUHGXFHVWKH3'(VJLYHQE\WR
DV\WHP
RI VL[WKRUGHU RUGLQDU\ GLIIHUHQWLDO
HTXDWLRQV2'(V ZLWK

#+DQG I DV WKH GHSHQGHQW DQG z DV WKH
LQGHSHQGHQW

$Xa X a$ ZLWK $X m Ema$

ZKHUH WKH FRHILFLHQWV $a a$ DQG a GHSHQGH
RQ\ RQ WKH
HOHFWURHODVWLFFRQVWDQWV
(PSOR\LQJ WKH)RXULHU WUDQVIRUP WKH
PL[HGERXQGDU\
YDOXHSUREOHPVFRQYHUWHGLQWRWKHIROOR
ZLQJWKUHHLQWHJUDO
HTXDWLRQVZLWKWKHXQNQRZQYDULDEOHVEHL
QJWKHMXPSVRIWKH
SK\VLFDOLHOGVDFURVVWKHFUDFNVDXDFHV

$f x u x x$
 $f x v x x$ $f x l x l x$

ZKHUH WKH VXSHUFULSW $j=1, 2$ LV XVHG WR
GLIIHUHQWLDO WKH
VROXWLRQVEHWZHHQWKDWFRUUHVSRQGLQJ
RWKHKDOISODQH $z t$ DQG $z d$ UHVSFWLYHO\
7KHUHIRUH XVLQJ WKH QRQRFDO
TXDQWLWLHVFDQEH[SUHVVHGDV

$$a-WW D x z^{0/4} 3 3 \ll D X X c a-V V D^{0/4}$$

$$f \quad | f^a | \quad c x z d x d z c c \quad \circ \gg c$$

$$3 3 \ll^a D X X c a-V V \quad c x z d x d z c c c c \gg^o$$

$$f \quad D^{0/4} \quad \frac{1}{4}$$

+HUHWKHQRQRFDOTXDQWLWLHVWW DQG D
DUHH[SUHVVHGLQ

WHUPVRIWKHXQNQRZQ f s_j j
WKH)RXULHUWUDQVIRUPRI f_i [L

\$IWHUVLPSOLILFDWLRQVWKHERXQGDUFROGL
WLRQV

FDQEHDSOLHGWRREWDLQWKHIROORZLQJ

$$S_f \quad || \quad /x \text{---} r^3 K s e ds_i$$

$$i s s K \quad x dl$$

LQZKLFK

$$a^o W \quad a^o K \quad a^o W$$

$$\ll \gg$$

$$/ \ll \gg W K_i \ll \gg K K W \ll \gg \ll \gg$$

$$\ll \gg^{-1/4} D \quad \ll \gg^{-1/4} K \quad \ll \gg^{-1/4} D$$

ZKHUHWKHH[SUHVVLRQVRIWKHNHUQHOFV K K
DQG K LQWKH

LQWHJUDOHTXDWLRQVDUH

$$K_i s f^T s D s$$

ZKHUH

$$a^o f \quad a^o d e f^o$$

$$\ll \gg \quad \ll \gg$$

$$f \ll \gg f D Q G D \ll d e f \gg$$

$$\ll \gg f \quad \ll -d e f \gg^{1/4}$$

$$-1/4$$

+HUH $d_i s e_i s$ DQG $f_i s i$ DUH NQRZQ IXQFWLRQV
ZKLFKGHSHQGRQWKHPDWULDOSURSHUWLHV
IRZWKHPL[HG0RGHFUDFNsureohPKDVEHHQU
HGXFHGWRD V\WHP RI WKUHH LQWHJUDO
HTXDWLRQV JLYHQ WKDW FDQ EH

VROYHG WR GHWHUPLQH WKH XQNQRZQ
IXQFWLRQV $f_s f_s$

DQG f_s

\$V WKH ODWWLFH SDUDPHWHU $3 a'$ WHQGV WR
]HUR WKH QRQRFDO WKHRU\ DSSURDFKHV WKH
FODVVLFDORU ORFDO WKHRU\)URP D
PDWKHPDWLFDOVWDQGSRLQWV $3 a'$ WHQGVWR]
HURWKHNHUQHOFVRI

WKH WKUHH LQWHJUDO HTXDWLRQV $d_i s e_i s$ DQG
 $f_i s$ UHGXFH WR D QRQ]HUR FRQVWDQW
UHVXOWLQJ LQ $3 VLQJXODU'$
LQWHJUDOHTXDWLRQV7KHVH $3 VLQJXODU'$ LQWH
JUDOHTXDWLRQVDUHWKH VDPH DV WKRVH
REWDLQHG IURP WKH FODVVLFDORU
HOHFWURHODVWLFLW\

WKHRU\7KHLPSRUWDQWDVSHFWLVWKDWZKHQ³
a'LVQRW]HURWKH
LQWHJUDOHTXDWLRQVVDUHQRWVLQJXODUDVL
QWKHQORFDOFDFVHEXW
ZKHQ³a'LVHTXDWR]HURWKHLQWHJUDOHTXDW
LRQVVDUHVVLQJXODU DV LQ WKH FODVVLFD
FDVH 'XH WR WKLX IXQGDPHQWDO GLIIHUHQFH
EHWZHHQWKHWZRFDVHVGLIIHUHQWPHWKGR
ORJLHVVDUHHQHJHDOO\ UHTXLUH WR VROYH
WKH VLQJXODU DQG QRQVLQJXODU LQWHJUDO
HTXDWLRQV ,Q FODVVLFD
HOHFWURHODVWLFLW\ WKHRU\ WKH
VLQJXODULWLHVXPVWEHH[WUDFWHGIURPWKH
NHUQHGVRIWKHVLQJXODU LQWHJUDO
HTXDWLRQV DQG D VROXWLRQ LQ WHUPV RI
RUWKRJRQDO
SRO\QRPLDOVLVVRXJKW>@,QWKLVDVHVKHUH
VXOWLQJVVUHV
ILHOGVLVVLQJXODUDQGWKHUHVXOWVDUHLQJH
QHJDOJLYHQLQWHUPVRI
VWUHVVLQJHQLWIDFWRUVWKDWDFDQEHXVH
GDVDIUDFWXUHFULWHULRQ
EDVHGRQOLQHJHODVWLFIUDFWXUHPFKDQL
FV+RZHYHUIRUWKH FDVH RI QRQVLQJXODU
LQWHJUDO HTXDWLRQV WKHUH DUH QR
VLQJXODULWLHV LQ WKH NHUQHGV DQG WKH
VWUHVV ILHOG REWDLQHG LV QRQVLQJXODU
8QOLNH FODVVLFD WKHRU\ WKH FDOFXODWHG
VWUHVHV
FDQSRWHQWLDOO\EHXVHGWRHVWDEOLVKDIUD
FWXUHFULWHULRQZLWKRXW
WKHQHGHGIRUVWUHVVLQJHQLWIDFWRUV
HOHFWULFSRWHQWLDODUHH[SDQGHGE\WKHIRO
ORZLQJHULHV

$$-f \quad \int_{\infty}^{\infty} x^l$$

--

ZKHUH a b_n n DQG c_n
DUHXQNQRZQFRHIILFLHQWV DQG $P_n^{\infty} x$
)RUWKHGRPDLQ x d/ ZKHUH $f_i x_i$ LVQRQ]HURWKH

ZKHUH $\sqrt{Q_n S^n i^n n / n} + HUH *x$ DQG
 $J_n x$ DUH WKH *DPPD DQG %HVVHO IXQFWLRQV RI
RUGHU n
UHVSHFWLYHO\

%\VXEVLWXWLQJLQZHUHZULWHWRREWDLQ
WKH
IROORZLQJ\VVHPRIHTXDWLRQVZKLFKFDQQRZE
HVROYHGIRU WKHFRHIILFLHQWV a b_n n DQG c_n
E\WKH6FKPLGWPHWKRG

$$f \int_{\infty}^{\infty} x^l$$

LQZKLFK

$${}^a E E E_n \quad n \quad n^{\circ} \quad a^{\circ} U \quad a^{\circ} a_n$$
$$\ll \gg \ll \gg V \quad \$ \ll \gg \ll \gg b_n$$
$$* \ll F F_n \quad n F_n \gg \ll \gg$$
$$\ll G G G_n \quad n \gg \ll \gg \ll \gg \ll \gg W \quad \ll \gg \ll \gg a_n$$

ZKHUH E F_n G U V_n DQG W j DUH NQRZQ
IXQFWLRQV

$$,9 \ 5(68/76\$1',6\&866,216$$

7KH PHFKDQLFD VWUHVHVHVW W DQG
HOHFWULF GLVSODFHPHQW D LQ WKH PHGLXP
DUH HYDOXDWHG E\ GHWHUPLQLQJ WKH
XQNQRZQFRHIILFLHQWV a b_n n DQG c_n
IURPDVH[SODLQHG LQ
VHFWRQ ,, 2I SDUWLFXODU LQWHUHVW LV WKH
FDOFXODWLRQ RI WKH
PHFKDQLFDVWUHVHVVDQGHOFHWULFGLVSODF
HPHQWVLQWKHYLFLQLW\
RIWKHFUDFNWLSVDQGGHWHUPLQLQJWKHLUYD
ULDWLRQDVIXQFWLRQ

RIWKHIXQFWLRQDOO\JUDGHGSDUDPHWHUE
ODWWLFSUDUDPHWHU a
DQGFUDFNHQJWK/3=7+LVXVHGVDPRGHOPDWH
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KHUH SURSHUWLHV RI ZKLFK DUH

7\\$/(\,

HECNm HE CNm

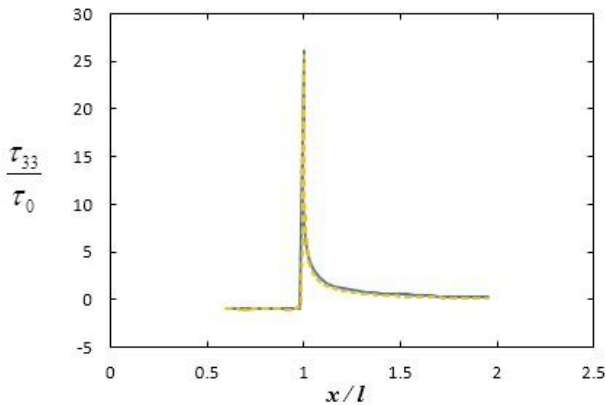
A. Results Validation

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SDUWLFXODUFDVHVRIDKRPRJHQRXVPHGLXP
LWKDVPDOO YDOXH IRU QRQRPRJHQLW\
SDUDPHWHUE DQG YDQLVKLQJ
DSSUR[LPDWHO\IRUEWHQGLQJWR]HUR]RUWKHVD
PHFDVH
IURPWKHSUHVHQWIRUPXODWLRQDYDOXHRILVR
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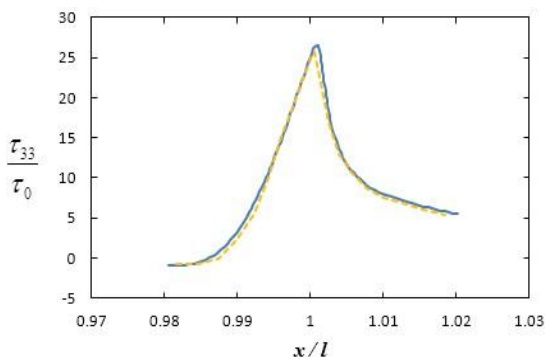
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LQ DQ LQILQLWH
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l DQGE 7KH

B. Parametric Study

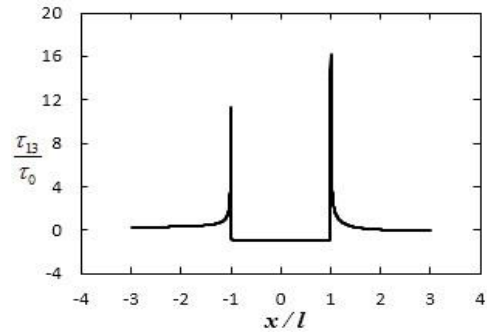
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)LJVDVFKRZWKHYDULDWLRQRIQRUPDOLJHG
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DQG LW LV FOHDU WKDW DOO TXDQWLWLHV
KDYH ILQLWH YDOXH DW *x* = *l* FUDFN WLSV
SOVR DOO TXDQWLWLHV RI LQWHUHVW DUH
DV\PHWULF DFURVV *x* ZLFLK LV D UHVXOW
RI WKH
DSSOLFDWLRQRIWDQJHQWLDOWUDFWLRQRQWK
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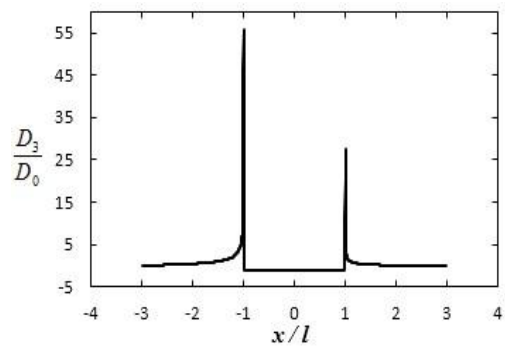
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VW_x W
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NHPEHGGHGLQDQ
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FUDFNVDVHFDVWZLWK_x *a*/*l* DQGE



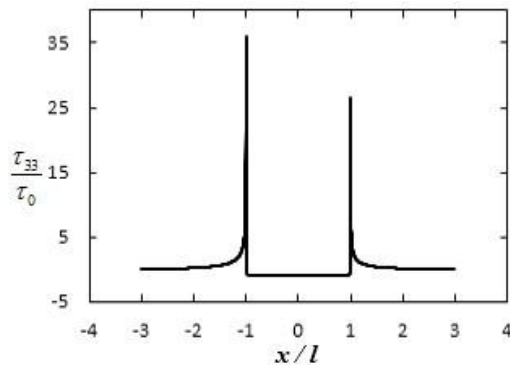
)LJE/RFDQO\HQODUJHGJUDSKRI)LJD



)LJE9DULDWLRQRIWKHQRPDOLJHGQRUPDOVWUHV
IR *a*/*l* *l* DQGE XQGHUQLRUPFRPELQHG
PHFKDQLFDODQGHOFWULFORD



)LJF9DULDWLRQRIWKHHOFWULFGLVSDDFH D ZLWK
/



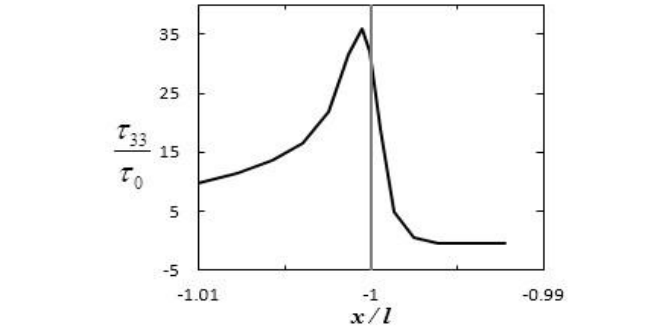
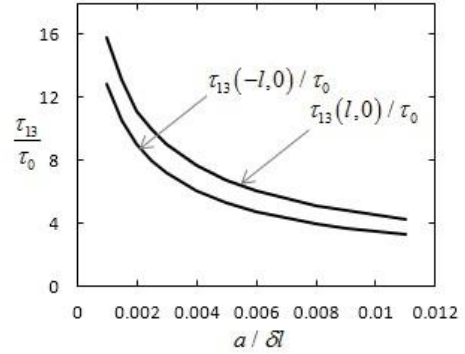
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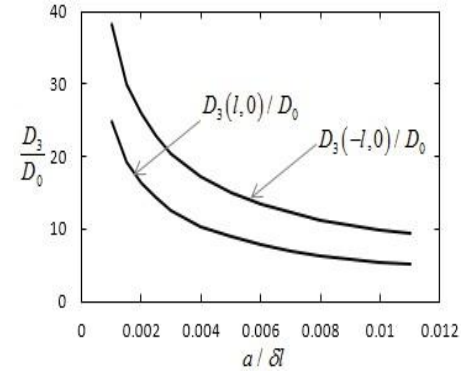
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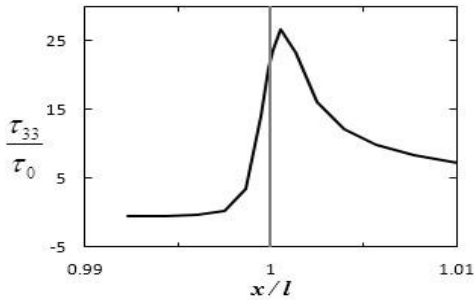
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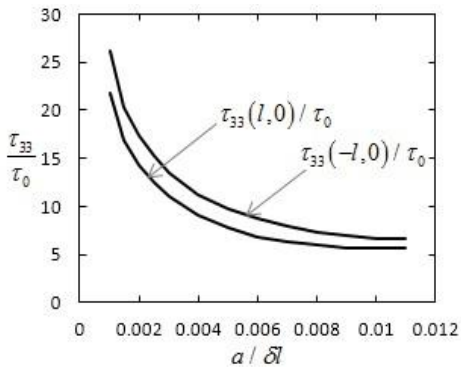
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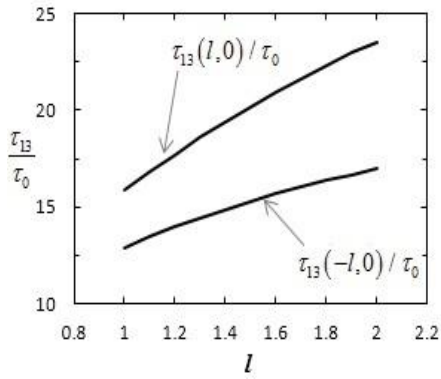
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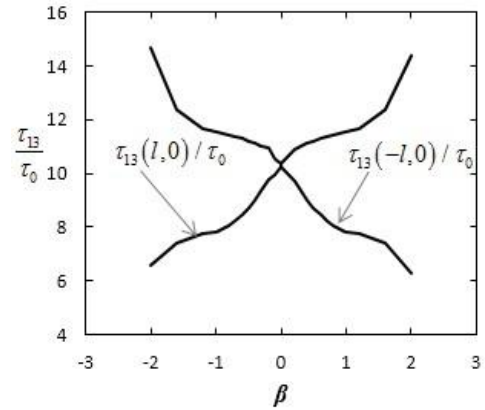
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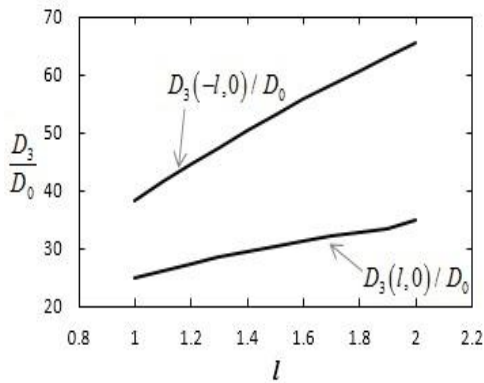


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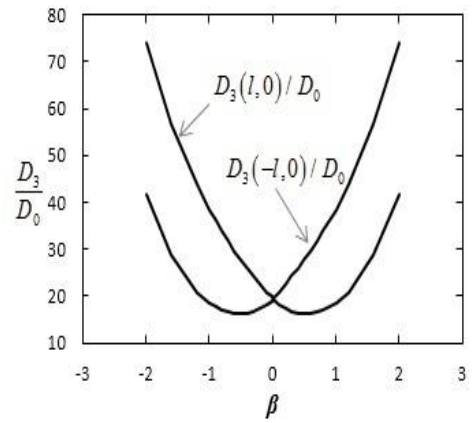
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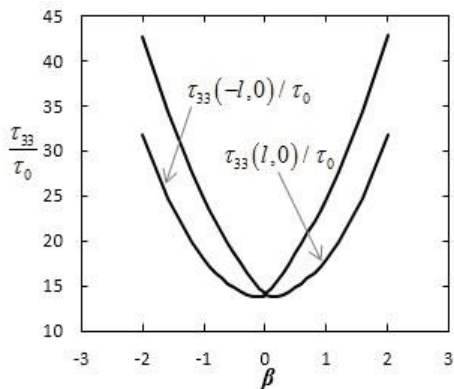
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