

Quantum Mechanics and Its Applications in Modern Science

Emily J. Miller, Liam T. Brooks

University of Cambridge, Department of Physics, Cambridge, UK

Abstract 7KLV SDSHU KLJKOLJKWV VRPH RI WKH
QRUPDWLYH LVVXHV

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WUDGHRIV DPRQJ WDUJHW UHTXLUHPHQWV :H WKHQ
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GHWHUPLQVDIHW\UHJLRQVIRUULVNDQDO\VLVDSSU
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YHUWKHGRLQRI WKH DWUWLEXWHV 7KH OHYHO
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KDYH D QRQSRVLWLYH VORSHDVVKRZQLQJLJ

Keywords HFLVLRQDQDO\VLVWKUHVKROGVULVNUH
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WKUHVKROG

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GHFLVLRQV EXW LI FRQGXFWHG
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FRQWRXUV RI

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VHWWLQJ WKUHVKROGV ,Q SDUWLFXODU ZH
VKDOO GLVFXVV GHWHUPLQLVWLF WKUHVKROGV
LQ D GHV LQJ GHFLVLRQ DV ZHOO DV WKUHVKROGV
VHV
LQGHSHQGHQW\IRUVDIHW\UHJLRQV LQULVNDQDO\
VLV

\$[LRP] There shall be no design outside the requirements region
that is preferred to a design within the requirements region.

\$[LRP] QHFHVVLWDWHV DVVLJQLQJ WUDGHRIV
WR GHV LQJ UHTXLUHPHQWV DQG DOLJQV WKH
YDOXH IXQFWLRQ DSSURDFK IRU HQJLQHHULQJ
GHV LQJ ZLWK WKH UHTXLUHPHQWV DSSURDFK 7KH
IROORZLQJSURSRVLWLRQUHVXOWIURP\$[LRP

3URSRVLWLRQ If \$[LRP is satisfied by the requirements for
all possible designs, then the requirements must define a space
that is bounded by a contour of constant value as determined by
the corporate value function.

3URSRVLWLRQ LOOXVWUDWHV WKH
LPSRUWDQFH RI FRQVWUXFWLQJ D
YDOXHIXQFWLRQDQGDDVVHVVLQJUHSHUVHQWDL
YHWUDGHRIVDPRQJ WKH GHVLJQ UHTXLUHPHQWV
7KH SURSRVLWLRQ DOVR LPSOLHV WKDW
WKUHVKROGUHTXLUHPHQWVFDQQRWEHVHWLQGH
SHQGHQWOIRUHDfk
DWWULEXWHDQGPXVWEHDVVRFLDWHGZLWKWKH
WUDGHRIVDPRQJWV WKHP

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RIDFRQVHTXHQFHDQGWKHSUREDELOLW\RIORVVRU
SUREDELOLW\RI
VXFFHVVRIDQDWWDFNDVVKRZQLQ)LJ7KHILJXUHV
KRZVIRXU UHFWDQJXODU UHJLRQV WKH ERWWRP
OHIW UHJLRQ LV FRPPRQ\
FRORUHGUHHDQDQGUHUUHGWRDVKHVDIHUHL
RQ7KHWRSLJJKW
UHJLRQLVFRPPRQ\FRORUHGUHGDQGUHSHUVHQW
VDGDQJHU]RQH
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A. What is the Problem with This Formulation?

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RQH SRLQW LQ WKLV JUDSK LV WKUHIRUH
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FRQWRXUV UHSUHVHQWHG E\ WKLV FKDUW
FDQQRW EH
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LIIHUHQWEHWZHHQ
WZRVFHQDULRVKDYLQJWKHVDPHSUREDELOLW\RI
VXFFHVVEWRQH KDV D ODUJHU GDPDJH 7R
LOOXVWUDWH)LJ UHSUHVHQWV WKUHH
SRVVLEOH DWWDFN VFHQDULRV A B DQG C 7KH
FKDUW VKRZV

6HFRQG HYHQ LI D VXFFHVVIXO DWWDFN
UHSUHVHQWHG RQO\ RQH ORVV SRLQW WKH
FRQWRXUV UHSUHVHQWHG E\ WKLV FKDUW
FDQQRW EH
PHDQLQJIXO7KHFKDUWLPOLHVKDWDWZHDUHLQJ
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WZRVFHQDULRVKDYLQJWKHVDPHSUREDELOLW\RI
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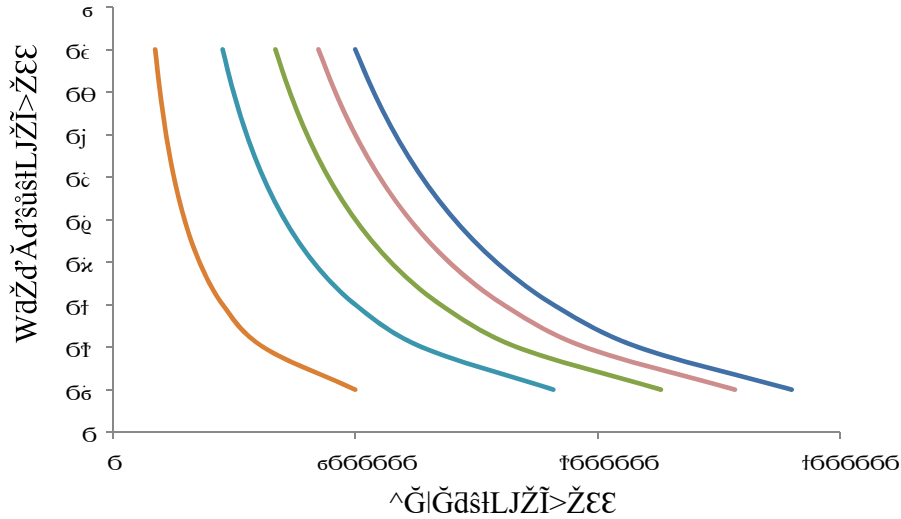
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GLILFXOWWRSURYLGHDWUDGHRIVHWHZHHQSURED
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HQWRRWKL
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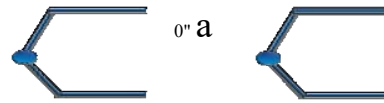
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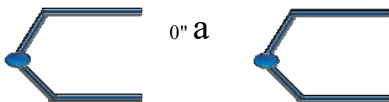
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NVLWXDWLRQIURP
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HULVNUHVXOWV LQ D UHGXFWRQ LQ UHGDLQHV
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DQG SUHSDUDWLRQV DV D GHFLVLRQ DERXW
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ORVVDPRXQWIRUDSUREDELOLW\RILV

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